

DIFFUSION DETERMINES THE RECURRENT GRAPH

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ABSTRACT. We consider diffusion on discrete measure spaces as encoded by Markovian semigroups arising from weighted graphs. We study whether the graph is uniquely determined if the diffusion is given up to order isomorphism. If the graph is recurrent then the complete graph structure and the measure space are determined (up to an overall scaling). As shown by counterexamples this result is optimal. Without the recurrence assumption, the graph still turns out to be determined in the case of normalized diffusion on graphs with standard weights and in the case of arbitrary graphs over spaces in which each point has the same mass.

These investigations provide discrete counterparts to studies of diffusion on Euclidean domains and manifolds initiated by Arendt and continued by Arendt / Biegert / ter Elst and Arendt / ter Elst.

A crucial step in our considerations shows that order isomorphisms are actually unitary maps (up to a scaling) in our context.

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INTRODUCTION

A famous question of M. Kac asks: *Can one hear the shape of a drum?* [35]. There is a substantial amount of research concerning this question over quite some time. Indeed, from Milnor's counterexample in 16 dimensions [48] over Sunada's general method [52] it took quite a while till a counterexample in two dimensions was given by Gordon / Webb / Wolpert in [21]. Of course,

the question may also be addressed in a discrete setting, i.e., for graphs. However, there one can easily find counterexamples as has been known for quite some time, see e.g. the textbook [12].

In mathematical terms the question of Kac concerns the eigenvalues of the Laplacian on a surface and whether a surface is determined by these eigenvalues. Thus, it asks whether two surfaces with unitarily equivalent Laplacians are naturally congruent. Of course, two Laplacians are unitarily equivalent if and only if their unitary groups are unitarily equivalent. In this sense, the question of Kac deals with unitary groups and unitary equivalence.

Recently, Arendt [2] studied the question: *Does diffusion determine the body?* Here, compared to the question of Kac, the unitary group is replaced by the diffusion semigroup and unitary equivalence is replaced by equivalence up to order isomorphism. The question then asks whether two relatively compact domains in Euclidean space with – up to order isomorphism – equal semigroups are naturally congruent. This question is then answered positively in [2] (under a weak regularity assumption on the domain). In fact, this can even be extended to manifolds, as shown by Arendt / Biegert / ter Elst in [4]. For compact manifolds an alternative treatment was also given by Arendt / ter Elst [5].

In the present paper we address this issue for diffusion on discrete measure spaces, i.e., for graphs. Our main result gives a positive answer to the question of Arendt in this setting. More specifically, our main result shows that

- diffusion determines the recurrent graph without killing and this result is optimal in a certain sense (see Theorem 7.1 and subsequent discussion).

When the underlying space is finite (corresponding to the compactness condition in [2, 5]) the recurrence assumption is trivially satisfied and we obtain an analogue to the results of [2, 5]. In fact, this can even be generalized to infinite graphs with finite total edge weight. This is studied in Section 8. In particular, we obtain that

- diffusion determines the infinite graph provided the total edge weight is finite, Corollary 8.5.

In all these situations diffusion determines the graph in the sense that it determines both the weights of the graphs and the measure on the underlying space. Let us emphasize that the weights and the measure are independent pieces of data. This is quite a difference to the framework of [2, 4, 5], where the measure is canonically fixed by the manifold structure.

Two most relevant situations in the graph setting actually come with a canonical measure. In these situations diffusion determines the graph without any recurrence or compactness assumption. Specifically, diffusion determines the graph

- if the measure of each point is one, Theorem 9.1, or,
- if the graph has standard weights (i.e., the edge weights take value in $\{0, 1\}$) and the Laplacian in question is the normalized Laplacian, Theorem 9.2.

These results can be considered as analogues to the results of [4].

Let us stress that these are particularly important situations. Indeed, for a long time most studies of Laplacians on graphs were concerned with these situations. In fact, it seems that the study of more general situations (as treated in the results above) has only become a focus of attention during the last five years or so. In this connection, we also point out some recent work on normalized Laplacians such as [6, 7, 9, 10, 27]. We also take this opportunity to mention the recent work [26], which introduces and studies discrete Laplacians in rather general geometric situations.

Along our way we also show that, without any recurrence or compactness/finiteness assumption, the diffusion always determines

- the combinatorial structure of the graph and the combinatorial distance (Theorem 5.1),
- the intrinsic pseudo metric ϱ introduced by Huang in [29] (Theorem 5.2).

While, obviously, our setting is quite different from [2, 4, 5], in terms of overall strategy our considerations certainly owe to [2, 4, 5]. In this context it seems worth pointing out some differences to these works. As for the results one main difference was already mentioned above: In our situation the weights and the measure on the underlying space are independent pieces of data (whereas the measure in the considerations of [2, 4, 5] is actually canonically given). As for the method, we do get some additional insights on the structure of intertwining order isomorphisms in our situation:

- Any intertwining order isomorphism is (up to a scaling) actually a unitary map (Corollary 3.5).
- In the case of regular diffusions the intertwining order isomorphisms are in one-to-one correspondence with generalized ground state transforms (Theorem 3.6 and Theorem 4.3).

The first point is rather remarkable as the categories of unitary maps and of order isomorphism are rather different in general. It can be seen as the main insight in the present paper. Indeed, in one way or other it is the crucial ingredient in the proofs of our theorems. On a more abstract level it can be understood as saying that existence of an order isomorphism is indeed a stronger requirement than existence of a unitary isomorphism and this can be seen in the context of explaining that order isomorphisms determine the graph while unitary isomorphisms do not.

In order to provide the proper setting for our results we first develop the theory of diffusion on discrete sets in quite some depth in Section 1. There, we basically follow [37, 38] (see [23, 24] as well). These works provide a framework for diffusion on discrete sets in terms of Dirichlet forms and graphs.

We then turn to studying how the graph structure is determined via diffusion. This is phrased in terms of order preserving maps intertwining the corresponding semigroups. In this context we first provide a general structure theorem on order preserving maps between ℓ^p spaces (Theorem 2.3) in Section 2. This result gives that any order isomorphism arises as a composition of a bijection τ and a scaling h . This is a variant of a Lamberti type theorem. It may be of independent interest. In Section 3, we then proceed

to study those order preserving isomorphisms which intertwine Laplacians on graphs. Here, we obtain a formula relating the weights of the different graphs in questions via the measures and the scaling h and the bijection τ coming from the order isomorphism (Theorem 3.6) and derive as a consequence that the scaling h is (super)harmonic (Corollary 3.9). On the technical level, these two results are the main tools of the paper. They are intimately linked to the observation in Corollary 3.5 that an intertwining order isomorphism is unitary (up to a scaling).

The connection to generalized ground state transforms and Theorem 4.3 are then the content of Section 4.

As a first application, we can then show in Section 5 how diffusion determines the graph structure and the pseudo metric ϱ .

A final ingredient in our reasoning is recurrence. This concept is discussed in Section 6. The results of that section are well-known. We mainly include the discussion in order to be self-contained.

Our main result, Theorem 7.1 is then stated and proven in Section 7. The discussion following that theorem shows the mentioned optimality of this result.

The particular case where the total edge weight is finite is considered in Section 8. In Section 9 we consider two situation of special interest, viz the case where the measure is normalized to be equal to one at each point and the situation where the measure is given by the degree and the weights take values in $\{0, 1\}$. In these cases, the graph turns out to be completely determined even without a recurrence assumption.

The theory of local Dirichlet spaces, i.e., topological spaces together with a strongly local Dirichlet form has been very successful in generalizing the situation of Riemannian manifolds. We consider our paper as a contribution to the emerging theory of discrete Dirichlet spaces, i.e., discrete spaces together with a Dirichlet form. For this reason we focus our attention on Dirichlet forms on graphs and the associated semigroups in continuous time. However, for graphs also semigroups in discrete time have been considered. It turns out that essential part of our results remain valid there as well. This is discussed in Section 10. We thank the anonymous referee for bringing up this point.

In order to make the paper self-contained we include an appendix on the intertwining property.

Part of the material presented in this paper is based on the Bachelor thesis of one of the authors (M. W.).

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1. GRAPHS AND DIFFUSION ON DISCRETE MEASURE SPACES

This section deals with diffusion on discrete sets. Here, diffusion is modeled by a Markovian semigroup. Such semigroups are in one-to-one correspondence to Dirichlet forms on the underlying space. Each such Dirichlet form gives naturally rise to a graph structure. In fact, regular Dirichlet forms

are even in one-to-one correspondence to graphs. Details are discussed below in this section. More specifically, we first introduce the necessary background on graphs, then turn to associated Dirichlet forms and afterwards discuss the associated Markovian semigroups. The main thrust of the paper is to study whether Markovian semigroups which are equivalent up to order isomorphism lead to the same graph structure. Along our way, we will also investigate certain metric features of the underlying graphs. The necessary background on metrics on graphs is discussed at the end of this section.

Throughout, we let X be a finite or infinitely countable set. All functions on X will be real valued.

Graphs. A pair (b, c) is called a *graph* over X if $b : X \times X \rightarrow [0, \infty)$ is symmetric, has zero diagonal, and satisfies

$$\sum_{y \in X} b(x, y) < \infty$$

for all $x \in X$ and $c : X \rightarrow [0, \infty)$ is arbitrary. The elements of X are then referred to as *vertices* and the function b is called the *edge weight* and c the *killing term*. Elements $x, y \in X$ are called *neighbors* and said to be *connected by an edge of weight $b(x, y)$* , if $b(x, y) > 0$. If the number of neighbors of each vertex is finite, then we call (b, c) or b *locally finite*. A finite sequence (x_0, \dots, x_n) of pairwise distinct vertices such that $b(x_{i-1}, x_i) > 0$ for $i = 1, \dots, n$ is called a *path* from x_0 to x_n . We say that the graph (b, c) is *connected* if, for every two vertices $x, y \in X$, there is a path from x to y .

Formal Laplacian and generalized forms. The space of real valued functions on X is denoted by $C(X)$ and the subspace of functions with finite support is denoted by $C_c(X)$. Throughout the characteristic function of a point $x \in X$ will be denoted as 1_x , i.e.,

$$1_x(y) = 1 \text{ for } x = y \text{ and } 1_x(y) = 0 \text{ otherwise.}$$

Given a graph (b, c) over X we introduce the associated *formal Laplacian* $\mathcal{L} = \mathcal{L}_{b,c}$ acting on

$$\mathcal{F} = \mathcal{F}_{b,c} = \{f \in C(X) : \sum_{y \in X} b(x, y)|f(y)| < \infty \text{ for all } x \in X\}$$

as

$$\mathcal{L}f(x) = \sum_{y \in X} b(x, y)(f(x) - f(y)) + c(x)f(x).$$

The operator \mathcal{L} can be seen as a discrete analogue of the Laplace Beltrami operator on a Riemannian manifold (and an additional potential).

Given a graph (b, c) over X , we furthermore define the *generalized form* $\mathcal{Q} = \mathcal{Q}_{b,c} : C(X) \rightarrow [0, \infty]$ by

$$\mathcal{Q}(f) := \frac{1}{2} \sum_{x, y \in X} b(x, y)|f(x) - f(y)|^2 + \sum_{x \in X} c(x)|f(x)|^2$$

and the *generalized form domain* by

$$\mathcal{D} = \mathcal{D}_{b,c} := \{f \in C(X) : \mathcal{Q}(f) < \infty\}.$$

Clearly, $C_c(X) \subseteq \mathcal{D}$ as $b(x, \cdot)$ is summable for every $x \in X$.

Since $\mathcal{Q}^{1/2}$ is a seminorm and satisfies the parallelogram identity

$$\mathcal{Q}(f+g) + \mathcal{Q}(f-g) = 2(\mathcal{Q}(f) + \mathcal{Q}(g)), \quad f, g \in \mathcal{D},$$

it gives, by polarization, a semi scalar product on \mathcal{D} via

$$\mathcal{Q}(f, g) = \frac{1}{2} \sum_{x, y \in X} b(x, y)(f(x) - f(y))(g(x) - g(y)) + \sum_{x \in X} c(x)f(x)g(x).$$

In the case when $c \not\equiv 0$ and b is connected, the form \mathcal{Q} defines a scalar product.

Obviously, \mathcal{Q} is *compatible with normal contractions* in the sense that

$$\mathcal{Q}(Cf) \leq \mathcal{Q}(f)$$

holds for any $f \in C(X)$ and any normal contraction $C : \mathbb{R} \rightarrow \mathbb{R}$. (Here, $C : \mathbb{R} \rightarrow \mathbb{R}$ is a *normal contraction* if both $|C(p)| \leq |p|$ and $|C(p) - C(q)| \leq |p - q|$ hold for all $p, q \in \mathbb{R}$.)

In [23, 24] an ‘integration by parts’ was shown that allows one to pair functions in \mathcal{F} and $C_c(X)$ via \mathcal{Q} . More precisely, the considerations of [23, 24] give for $f \in \mathcal{F}$ and $v \in C_c(X)$

$$\begin{aligned} & \frac{1}{2} \sum_{x, y \in X} b(x, y)(f(x) - f(y))(v(x) - v(y)) + \sum_{x \in X} c(x)f(x)v(x) \\ &= \sum_{x \in X} f(x)(\mathcal{L}v)(x) \\ &= \sum_{x \in X} (\mathcal{L}f)(x)v(x), \end{aligned}$$

where all sums converge absolutely. Moreover, it is shown there that \mathcal{D} is a subset of \mathcal{F} and that for $f \in \mathcal{D}$ the preceding sums all agree with $\mathcal{Q}(f, g)$. These formulae will be referred to as *Greens formulae*.

Dirichlet forms and their generators. We now assume that we are additionally given a measure m on X of full support. Then, suitable restrictions of \mathcal{Q} are in correspondence with certain self-adjoint restrictions of the formal operator \tilde{L} (defined below) on the Hilbert space $\ell^2(X, m)$ of real valued square summable functions equipped with the scalar product

$$\langle u, v \rangle = \langle u, v \rangle_m = \sum_{x \in X} u(x)v(x)m(x),$$

and norm $\|u\| = \|u\|_m = \sqrt{\langle u, u \rangle}$. This is discussed next:

Let (b, c) be graph over X and let $\mathcal{Q} = \mathcal{Q}_{b, c}$ be the associated form on the domain $\mathcal{D} = \mathcal{D}_{b, c}$ as discussed above. Let now Q be a closed non-negative form on $\ell^2(X, m)$, whose domain $D(Q)$ satisfies $C_c(X) \subseteq D(Q) \subseteq \mathcal{D} \cap \ell^2(X, m)$ and which satisfies

$$Q(u, v) = \mathcal{Q}(u, v),$$

for $u, v \in D(Q)$. Thus, Q is essentially a restriction of \mathcal{Q} . For such a form we define $Q(u) := Q(u, u)$ for $u \in D(Q)$ and $Q(u) := \infty$ for $u \notin D(Q)$. Such a form will be referred to as a *form associated to the graph (b, c) over (X, m)* . If such a form is a Dirichlet form (i.e., satisfies $Q(Cf) \leq Q(f)$ for all $f \in D(Q)$) it will be called a *Dirichlet form associated to the graph (b, c)* .

By general results on closed forms, any Q associated to a graph comes with a unique self-adjoint operator L such that $D(Q)$ is just the domain of definition of $L^{1/2}$ and $Q(f, g) = \langle L^{1/2}f, L^{1/2}g \rangle$ holds for all $f, g \in D(Q)$. The operator L is known as the *generator of the form* Q . We will also refer to it as *Laplacian corresponding to the form* Q .

By [24, Proposition 3.3] we have that the Laplacian L corresponding to a form Q associated to a graph satisfies

$$Lu = \tilde{L}u,$$

for all $u \in D(L)$. Here, \tilde{L} is just the operator $\frac{1}{m}\mathcal{L}$ acting as

$$\tilde{L}f(x) = \frac{1}{m(x)} \sum_{y \in X} b(x, y)(f(x) - f(y)) + \frac{c(x)}{m(x)}f(x).$$

There are two natural examples, $Q^{(N)}$ and $Q^{(D)}$, which we refer to as the *forms with Neumann* and *Dirichlet boundary conditions*, respectively. The form $Q^{(N)}$ has the domain

$$D(Q^{(N)}) = \mathcal{D} \cap \ell^2(X, m) = \{u \in \ell^2(X, m) : \mathcal{Q}(u) < \infty\}.$$

The form $Q^{(D)}$ has the domain

$$D(Q^{(D)}) = \overline{C_c(X)}^{\|\cdot\|_{\mathcal{Q}}},$$

where

$$\|u\|_{\mathcal{Q}} := (\mathcal{Q}(u) + \|u\|^2)^{\frac{1}{2}}.$$

We denote the corresponding self adjoint operators by $L^{(N)}$ and $L^{(D)}$.

By construction, both $Q^{(N)}$ and $Q^{(D)}$ are Dirichlet forms as \mathcal{Q} is compatible with normal contractions. In fact, this is clear for $Q^{(N)}$ and follows for $Q^{(D)}$ by a result of Fukushima (see [37] for discussion). Moreover, $Q^{(D)}$ is regular, i.e., $D(Q^{(D)}) \cap C_c(X)$ is dense in $D(Q^{(D)})$ with respect to $\|\cdot\|_{\mathcal{Q}}$ and in $C_c(X)$ with respect to $\|\cdot\|_{\infty}$.

The forms $Q^{(D)}$ and $Q^{(N)}$ are of particular relevance in the theory of forms associated to graphs in the sense that a symmetric, closed quadratic form Q is associated to the graph if and only if $Q^{(D)} \subseteq Q \subseteq Q^{(N)}$, see [19]. Here, we write $Q_1 \subseteq Q_2$ if $D(Q_1) \subseteq D(Q_2)$ and $Q_1(u) = Q_2(u)$ for $u \in D(Q_1)$.

Markovian semigroups. We will be interested in the semigroups associated to graphs. Here, we shortly sketch the necessary ingredients. For further discussion and proofs we refer to standard references like [17, 13].

Any form Q associated to a graph (b, c) over (X, m) is obviously non-negative. Thus, the corresponding Laplacian L gives rise to a semigroup e^{-tL} , $t \geq 0$. This semigroup provides the solution for the associated heat equation as follows. For any $u \in D(L)$ the function

$$\psi : [0, \infty) \rightarrow \ell^2(X, m), \quad t \mapsto e^{-tL}u,$$

is a solution of the heat equation with initial value u

$$(HE) \quad \frac{d}{dt}\psi_t = \frac{1}{m}\mathcal{L}\psi_t, \quad t > 0 \quad \psi_0 = u.$$

In fact, it is possible to describe the form Q in terms of the associated semigroup as follows:

$$D(Q) = \{f \in \ell^2(X, m) : \lim_{t \downarrow 0} \frac{1}{t} \langle f - e^{-tL} f, f \rangle < \infty\}$$

$$Q(f, g) = \lim_{t \downarrow 0} \frac{1}{t} \langle f - e^{-tL} f, g \rangle.$$

We will be interested in semigroups e^{-tL} , $t \geq 0$, which are *Markovian*, i.e., satisfy $0 \leq e^{-tL} f \leq 1$ for all $t \geq 0$ whenever $f \in \ell^2(X, m)$ satisfies $0 \leq f \leq 1$. Indeed, Markovian semigroups are the natural candidates for modeling diffusion processes. By the second Beurling Deny criteria the semigroup is Markovian if and only if the form Q is a Dirichlet form.

If e^{-tL} is a Markovian semigroup, then it leaves the space $\ell^2(X, m) \cap \ell^p(X, m)$ invariant and its restriction to this space can be extended uniquely to a strongly continuous semigroup on $\ell^p(X, m)$ for any $p \in [1, \infty)$. The generator of this semigroup will be denoted by $L^{(p)}$ and the semigroup will be denoted by $e^{-tL^{(p)}}$. It provides a solution of the heat equation (HE) with initial data $u \in \ell^p(X, m)$.

The preceding discussion shows that graphs and the associated Dirichlet forms naturally encode diffusion on discrete sets.

Intrinsic metrics. We finish this section with a discussion of certain metrics on discrete sets arising from graphs. These metrics will turn out to be invariant under the order isomorphism considered later. Recall that a *pseudo metric* on the set X satisfies all properties of a metric except that it may be degenerate (i.e., vanish outside the diagonal). Subsequently we will consider metrics only for connected graphs. However, the case of general graphs could easily be included e.g. by restricting attention to connected components and declaring the distance between different components to be infinity.

Definition 1.1 (Combinatorial metric). Let (b, c) be a connected graph over X . Then, the metric d defined by

$$d(x, y) = \inf_{\gamma} \sharp \gamma,$$

where the infimum is taken over all paths $\gamma = (x_0, \dots, x_n)$ connecting x and y and $\sharp \gamma = n$ is the number of edges in γ , is called *combinatorial graph metric*.

The following definition of intrinsic metrics gives a class of (pseudo) metrics of graphs that allow one to study spectral geometry on graph in a similar way as on Riemannian manifolds. Such metrics were only very recently brought forward and studied systematically for arbitrary regular Dirichlet forms in [16] (see [53, 14] for related material as well). They have already proven quite useful in the study of graphs [8, 11, 19, 22, 28, 29, 31, 25, 33, 43]. While [16] deals with general regular Dirichlet forms, here we only present the definition for graphs.

Definition 1.2 (Intrinsic metric for graphs [16]). Let (b, c) be a graph over (X, m) . Then, a pseudo metric δ on X is said to be *intrinsic* if

$$\sum_{y \in X} b(x, y) \delta(x, y)^2 \leq m(x), \quad x \in X$$

Remarks. A few remarks on this definition are in order. To simplify the discussion we only consider the case $c = 0$.

(a) We can write

$$\begin{aligned} \mathcal{Q}(f, f) &= \frac{1}{2} \sum_{x \in X} \left(\frac{1}{m(x)} \sum_{y \in X} b(x, y) |f(x) - f(y)|^2 \right) m(x) \\ &= \frac{1}{2} \sum_{x \in X} m(x) \|\nabla f(x)\|^2 \end{aligned}$$

with

$$\|\nabla f(x)\| = \sqrt{\frac{1}{m(x)} \sum_{y \in X} b(x, y) |f(x) - f(y)|^2}.$$

This suggests to single out the set of functions \mathcal{A} defined via

$$\mathcal{A} := \{f \in C(X) : \|\nabla f(x)\| \leq 1 \text{ for all } x \in X\}.$$

Then, a metric δ can easily be seen to be intrinsic if and only if

$$\text{Lip}_\delta^1 \subset \mathcal{A},$$

where Lip_δ^1 is the set of Lipschitz functions with constant one with respect to δ .

(b) The set \mathcal{A} is in general not closed under taking suprema: Consider e.g. the subgraph of \mathbb{Z} given by the integers from -1 to 1 , i.e.,

$$X = \{-1, 0, 1\}, \quad m \equiv 1$$

and

$$b(-1, 0) = b(0, -1) = b(1, 0) = b(0, 1) = 1, \quad b(1, -1) = b(-1, 1) = 0.$$

Let f_+ be the characteristic function of 1 and f_- the characteristic function of -1 . Clearly, f_\pm belong to \mathcal{A} . On the other hand a direct calculation shows that $f = f_+ + f_- = \max\{f_+, f_-\}$ does not belong to \mathcal{A} (see an example in [16] for a similar reasoning as well).

(c) The set Lip_δ^1 is closed under taking suprema by general principles. By (b), the set Lip_δ^1 will then in general not be equal to \mathcal{A} . Put differently, in general

$$\sigma(x, y) := \sup\{|f(x) - f(y)| : f \in \mathcal{A}\}$$

will not be an intrinsic metric. This is very different for strongly local Dirichlet forms; compare the Rademacher type theorem in [16] and the considerations of [51].

(d) Define for $\alpha > 0$ the scaled version of the combinatorial metric via $d_\alpha = \alpha d$. Obviously, d_α is then an intrinsic metric if and only if

$$\alpha \sum_{y \in X} b(x, y) \leq m(x),$$

for all $x \in X$. On the other hand, by a result of [24], boundedness of the associated Laplacian is equivalent to existence of a $C > 0$ such that

$$\frac{1}{C} \sum_{y \in X} b(x, y) \leq m(x)$$

for all $x \in X$. Thus, (a suitably scaled version of) the combinatorial metric d is an intrinsic metric if and only if the associated Laplacian is bounded (compare [16, 25] as well).

A specific example of an intrinsic pseudo metric is the pseudo metric ϱ introduced by Huang in [29], Lemma 1.6.4., for connected graphs and defined by

$$\begin{aligned} \varrho : X \times X &\rightarrow [0, \infty), \\ \varrho(x, y) &= \inf_{\gamma} \sum_{k=1}^n \min \left\{ \frac{1}{\sqrt{\text{Deg}(x_{k-1})}}, \frac{1}{\sqrt{\text{Deg}(x_k)}} \right\}. \end{aligned}$$

Here, the infimum is taken over all paths $\gamma = (x_0, \dots, x_n)$ connecting x and y and the *generalized degree* Deg is the function

$$\text{Deg} : X \rightarrow [0, \infty), \quad x \mapsto \frac{1}{m(x)} \left(\sum_{y \in X} b(x, y) + c(y) \right).$$

Remark. In general ϱ is not a metric. However, if the graph is locally finite then, clearly, any point has a positive distance to each of its neighbors and ϱ is a metric.

2. STRUCTURE OF ORDER ISOMORPHISMS BETWEEN ℓ^p -SPACES

In this section we deal with positive operators between ℓ^p -spaces. We present a structure theorem on order isomorphism between ℓ^p spaces, Theorem 2.3.

Definition 2.1 (Order isomorphism). Let $(\Omega_1, \mathcal{A}_1, \mu_1)$, $(\Omega_2, \mathcal{A}_2, \mu_2)$ be arbitrary measure spaces and $U : L^p(\Omega_1, \mu_1) \rightarrow L^p(\Omega_2, \mu_2)$ be linear. Then, U is called *order preserving* if $Uf \geq 0$ whenever $f \geq 0$. The map U is called an *order isomorphism* if U is invertible and both U and its inverse are order preserving.

We start with an easy technical lemma, which is used in the proof of Theorem 2.3.

Lemma 2.2. *Let $(\Omega_1, \mathcal{A}_1, \mu_1)$, $(\Omega_2, \mathcal{A}_2, \mu_2)$ be arbitrary measure spaces and $U : L^p(\Omega_1, \mu_1) \rightarrow L^p(\Omega_2, \mu_2)$ an order isomorphism. Let $f, g \in L^p(\Omega_1, \mu_1)$ be given such that $fg = 0$. Then $(Uf)(Ug) = 0$.*

Proof. If $h \leq f, g$ then $Uh \leq Uf, Ug$ and $h' \leq h$ implies $Uh' \leq Uh$ since U is an order isomorphism. We conclude $U(f \wedge g) = Uf \wedge Ug$ and $U(f \vee g) = Uf \vee Ug$. With $f^+ = 0 \vee f$ and $f^- = -f \vee 0$ we obtain $U|f| = |Uf|$.

Furthermore, we have $fg = 0$ if and only if μ_1 -almost everywhere $f(x) = 0$ or $g(x) = 0$ holds, that is, $|f(x)| \wedge |g(x)| = 0$ for μ_1 -almost all $x \in \Omega_1$. Hence, $fg = 0$ if and only if $|f| \wedge |g| = 0$.

It follows that $|Uf| \wedge |Ug| = U(|f| \wedge |g|) = 0$ and hence $(Uf)(Ug) = 0$. \square

The following theorem is a variant of the Banach-Lamberti type result (cf. [44], Theorem 3.1). Note that we work with an order preserving transformation U instead of isometries. This gives us the advantage that this theorem is valid also for $p = 2$.

Theorem 2.3. *Let (X_1, m_1) , (X_2, m_2) be discrete measure spaces, $p \in [1, \infty)$ and $U: \ell^p(X_1, m_1) \rightarrow \ell^p(X_2, m_2)$ an order isomorphism. Then there exists a unique function $h: X_2 \rightarrow (0, \infty)$ and a unique bijection $\tau: X_2 \rightarrow X_1$ such that $Uf = h \cdot (f \circ \tau)$ for all $f \in \ell^p(X_1, m_1)$.*

Proof. Uniqueness of h and τ is clear. It remains to show existence. For $y \in X_2$ define $S_y: \ell^p(X_1, m_1) \rightarrow \mathbb{R}$, $S_y(f) = Uf(y)$. Since S_y is positive, it is a continuous functional due to standard results on positive operators, see e.g. [1, Theorem 12.3].

By the representation theorem for the dual of ℓ^p there is a function $g_y \in \ell^q(X_1, m_1)$, $\frac{1}{p} + \frac{1}{q} = 1$, such that $S_y(f) = \sum_{\xi \in X_1} f(\xi)g_y(\xi)m_1(\xi)$ for all $f \in \ell^p(X_1, m_1)$.

Since U is surjective, $\text{supp } g_y$ is non-empty. Now let $x_0 \in \text{supp } g_y$ and $x \neq x_0$. Then we have $1_{x_0}1_x = 0$ and by Lemma 2.2 we obtain

$$0 = U1_x(y) \cdot U1_{x_0}(y) = g_y(x)m_1(x)g_y(x_0)m_1(x_0).$$

Since m_1 has full support, we have $g_y(x) = 0$. Thus, $\text{supp } g_y = \{x_0\}$.

Hence, there is a function $h: X_2 \rightarrow \mathbb{R}$ and a map $\tau: X_2 \rightarrow X_1$ such that $Uf(y) = h(y)f(\tau(y))$ for all $f \in \ell^p(X_1, m_1)$, where $\text{supp } g_y = \{\tau(y)\}$ and

$$h(y) = U1_{\tau(y)}(y) = g_y(\tau(y))m_1(\tau(y)).$$

By positivity of U we get $h(y) \geq 0$ for all $y \in X_2$.

Analogously we obtain a function $\tilde{h}: X_1 \rightarrow [0, \infty)$ and a map $\tilde{\tau}: X_1 \rightarrow X_2$ such that $U^{-1}g = \tilde{h} \cdot g \circ \tilde{\tau}$ for all $g \in \ell^2(X_2, m_2)$. Hence, we have for all $y \in X_2$

$$\begin{aligned} 1 &= 1_y(y) = UU^{-1}1_y(y) \\ &= (h \cdot (\tilde{h} \cdot 1_y \circ \tilde{\tau}) \circ \tau)(y), \\ &= \begin{cases} h(y)\tilde{h}(\tau(y)) & : \tilde{\tau}(\tau(y)) = y \\ 0 & : \text{else,} \end{cases} \end{aligned}$$

that is, $\tilde{\tau} \circ \tau = \text{id}_{X_2}$ and $h(y) \neq 0$ for all $y \in X_2$. The equation $\tau \circ \tilde{\tau} = \text{id}_{X_1}$ is proven analogously. Thus, τ is a bijection with inverse $\tilde{\tau}$. \square

Remark. If we think of elements of ℓ^p as sequences and operators between ℓ^p -spaces as infinite matrices, the above theorem shows that the matrix of an order isomorphism has exactly one non-zero entry in every row and column, which is strictly positive.

Definition 2.4 (Bijection and scaling associated to order isomorphism). Let $U: \ell^p(X_1, m_1) \rightarrow \ell^p(X_2, m_2)$ be an order isomorphism and $h: X_2 \rightarrow (0, \infty)$ and $\tau: X_2 \rightarrow X_1$ the unique functions with $Uf = h \cdot (f \circ \tau)$ for all $f \in \ell^p(X_1, m_1)$. Then, τ is called the *bijection associated to U* and h is called the *scaling associated to U* .

For later use we now compute an explicit formula for the adjoint of an order isomorphism.

Lemma 2.5. *Let $p \in [1, \infty)$ be given and let q with $\frac{1}{p} + \frac{1}{q} = 1$ be chosen. Let $\tau: X_2 \rightarrow X_1$ be a bijection and $h: X_2 \rightarrow (0, \infty)$ such that for all $f \in \ell^p(X_1, m_1)$ the function $h \cdot (f \circ \tau)$ belongs to $\ell^p(X_2, m_2)$. Define $U: \ell^p(X_1, m_1) \rightarrow \ell^p(X_2, m_2)$, $f \mapsto h \cdot (f \circ \tau)$. Then the adjoint of U is given by*

$$U^*: \ell^q(X_2, m_2) \rightarrow \ell^q(X_1, m_1), U^*f = \frac{1}{m_1}(h \cdot f \cdot m_2) \circ \tau^{-1}.$$

Moreover, for all $f \in C_c(X_1, m_1)$ and all $g \in C_c(X_2, m_2)$ we have

$$\begin{aligned} U^*Uf(x) &= \frac{m_2(\tau^{-1}(x))}{m_1(x)} h(\tau^{-1}(x))^2 f(x) \\ UU^*g(y) &= \frac{m_2(y)}{m_1(\tau(y))} h(y)^2 g(y). \end{aligned}$$

Proof. The operator U is obviously closed and defined on the whole space. Thus, it is a bounded operator. For this reason its adjoint is a bounded operator as well. Denoting the dual pairing between ℓ^p and ℓ^q by (\cdot, \cdot) we can now compute for any $g \in \ell^q(X_2, m_2)$

$$\begin{aligned} U^*g(x) &= (U^*g, \frac{1}{m_1(x)}1_x) \\ &= (g, U(\frac{1}{m_1(x)}1_x)) \\ &= \sum_{y \in X_2} g(y)h(y) \frac{1}{m_1(x)}1_x(\tau(y))m_2(y) \\ &= g(\tau^{-1}(x))h(\tau^{-1}(x)) \frac{m_2(\tau^{-1}(x))}{m_1(x)}. \end{aligned}$$

The second assertion follows easily:

$$\begin{aligned} U^*Uf(x) &= \frac{1}{m_1(x)}(h \cdot Uf \cdot m_2) \circ \tau^{-1}(x) \\ &= \frac{((h^2 \cdot m_2) \circ \tau^{-1})(x)}{m_1(x)} \cdot f(x). \end{aligned}$$

A similar computation for UU^* finishes the proof. \square

3. STRUCTURE OF ORDER ISOMORPHISMS INTERTWINING GRAPHS

In Section 1 we have seen that graphs naturally give rise to Markovian semigroups. We are interested in graphs with semigroups which are equal up to order isomorphism. This is captured in the concept of intertwining order isomorphism. In this section, we first discuss and characterize this intertwining property and then study the functions h and τ arising from intertwining order isomorphisms U . We will see that these functions behave well with respect to the corresponding edge weights. On the technical level this is the core of the paper.

The following proposition gives a characterization of intertwining of operators in terms of semigroups. It is certainly well known. For the convenience

of the reader we provide a proof in the appendix in Proposition A.1 (compare the appendix of [41] for related material as well).

Proposition 3.1. *For $i = 1, 2$, let Q_i be a Dirichlet form associated to the graph (b_i, c_i) over (X_i, m_i) . Let $p \in [1, \infty)$ be given and consider the associated semigroups on $\ell^p(X_i, m_i)$ given by $e^{-tL_i^{(p)}}$, $i = 1, 2$. Furthermore, let $U: \ell^p(X_1, m_1) \rightarrow \ell^p(X_2, m_2)$ be an order isomorphism. Then the following assertions are equivalent:*

- (i) $Ue^{-tL_1^{(p)}} = e^{-tL_2^{(p)}}U$ for all $t \geq 0$.
- (ii) $UD(L_1^{(p)}) = D(L_2^{(p)})$ and $UL_1^{(p)}f = L_2^{(p)}Uf$ for all $f \in D(L_1^{(p)})$.

Definition 3.2 (Intertwining of operators). Consider the situation of the proposition. The order isomorphism U is said to *intertwine* $L_1^{(p)}$ and $L_2^{(p)}$ if one of the equivalent assertions of the previous proposition holds.

Lemma 3.3. *For $i = 1, 2$, let Q_i be a Dirichlet form associated to the graph (b_i, c_i) over (X_i, m_i) and assume (b_1, c_1) is connected. Let $p \in [1, \infty)$ be given and consider the semigroups associated to Q_i on $\ell^p(X_i, m_i)$ given by $e^{-tL_i^{(p)}}$ for $i = 1, 2$. Let $U: \ell^p(X_1, m_1) \rightarrow \ell^p(X_2, m_2)$ be an order isomorphism intertwining $L_1^{(p)}$ and $L_2^{(p)}$ with associated scaling h and bijection τ . Then, the function*

$$X_1 \rightarrow (0, \infty), x \mapsto \frac{m_2(\tau^{-1}(x))}{m_1(x)} h(\tau^{-1}(x))^2$$

is constant.

Proof. We denote the function $x \mapsto \frac{m_2(\tau^{-1}(x))}{m_1(x)} h(\tau^{-1}(x))^2$ by φ and write (\cdot, \cdot) for the dual pairing between ℓ^p and ℓ^q , with $\frac{1}{p} + \frac{1}{q} = 1$. Lemma 2.5 shows

$$\begin{aligned} \varphi(y)(e^{-tL_1^{(p)}} 1_x, 1_y) &= (e^{-tL_1^{(p)}} 1_x, U^*U 1_y) \\ &= (Ue^{-tL_1^{(p)}} 1_x, U 1_y) \\ &= (e^{-tL_2^{(p)}} U 1_x, U 1_y). \end{aligned}$$

As semigroups of Dirichlet forms act on all ℓ^p spaces, coincide on their intersections and are self-adjoint on ℓ^2 , the above implies

$$\begin{aligned} \varphi(y)\langle e^{-tL_1^{(2)}} 1_x, 1_y \rangle &= \langle e^{-tL_2^{(2)}} U 1_x, U 1_y \rangle \\ &= \langle e^{-tL_2^{(2)}} U 1_y, U 1_x \rangle \\ &= \varphi(x)\langle e^{-tL_1^{(2)}} 1_y, 1_x \rangle \\ &= \varphi(x)\langle e^{-tL_1^{(2)}} 1_x, 1_y \rangle. \end{aligned}$$

Semigroups associated with connected graphs are positivity improving, that is $\langle e^{-tL_1^{(2)}} 1_x, 1_y \rangle > 0$ for all $x, y \in X_1$ (see e.g. Corollary 2.9 in [37]). Dividing by this quantity finishes the proof. \square

Lemma 3.4. *For $i = 1, 2$, let Q_i be a Dirichlet form associated to the graph (b_i, c_i) over (X_i, m_i) and assume (b_1, c_1) is connected. Let $p \in [1, \infty)$ be given and consider the semigroups associated to Q_i on $\ell^p(X_i, m_i)$ given*

by $e^{-tL_i^{(p)}}$ for $i = 1, 2$. Let $U: \ell^p(X_1, m_1) \rightarrow \ell^p(X_2, m_2)$ be an order isomorphism intertwining $L_1^{(p)}$ and $L_2^{(p)}$ with associated bijection τ and associated scaling h . Then there exists a constant $\beta > 0$ such that the operator $\tilde{U}: \ell^2(X_1, m_1) \rightarrow \ell^2(X_2, m_2)$, $f \mapsto h \cdot (f \circ \tau)$ satisfies

$$\|f\|_{m_1}^2 = \beta \|\tilde{U}f\|_{m_2}^2.$$

Furthermore, \tilde{U} is an order isomorphism intertwining $L_1^{(2)}$ and $L_2^{(2)}$.

Proof. Since \tilde{U} and U as well as $e^{-tL_i^{(p)}}$ and $e^{-tL_i^{(2)}}$ coincide on $\ell^2 \cap \ell^p$, it suffices to show the continuity statement on \tilde{U} and its surjectivity. Using Lemma 3.3 we set

$$\beta = \frac{m_1(x)}{m_2(\tau^{-1}(x))h(\tau^{-1}(x))^2},$$

for some $x \in X_1$. Then an easy direct computation shows

$$\|f\|_{m_1}^2 = \beta \|\tilde{U}f\|_{m_2}^2,$$

for each $f \in \ell^2(X_1, m_1)$. Thus, \tilde{U} is an isometry (up to a scaling). Moreover, \tilde{U} obviously maps $C_c(X_1)$ onto $C_c(X_2)$. As C_c is dense in ℓ^2 and \tilde{U} is an isometry (up to a scaling), \tilde{U} is in fact surjective. This finishes the proof. \square

Remark. Lemma 3.4 shows that intertwining on ℓ^p always implies intertwining on ℓ^2 . Hence, for a better readability of the statements we subsequently only formulate theorems for the case $p = 2$. The reader however should keep in mind that the theory remains true in the general case $p \in [1, \infty)$. We also drop the superscript on the ℓ^2 generator of a Dirichlet form, i.e., write L instead of $L^{(2)}$ (as already discussed in the first section).

Corollary 3.5. *Assume the situation of the previous lemma. Then the operator $\sqrt{\beta}\tilde{U}: \ell^2(X_1, m_1) \rightarrow \ell^2(X_2, m_2)$ is unitary.*

Proof. By the previous lemma the operator $\sqrt{\beta}\tilde{U}$ is a surjective isometry and hence unitary. \square

Remark. The previous corollary provides a crucial ingredient of the investigations of this paper. Indeed, it is underlying the proof of the next theorem, which is the main technical tool in our considerations. On the abstract level it is remarkable in connecting the – a priori rather different – concepts of unitary isomorphisms and order isomorphisms in our context (compare discussion in the introduction).

We can now come to the main technical result of the paper. Indeed, the subsequent theorem gives the main connection between the weights of graphs whose Laplacians are intertwined by an order preserving isomorphism.

Theorem 3.6. *For $i = 1, 2$, let Q_i be a Dirichlet form associated to the graph (b_i, c_i) over (X_i, m_i) such that (b_1, c_1) is connected. Let $U: \ell^2(X_1, m_1) \rightarrow \ell^2(X_2, m_2)$ be an order isomorphism intertwining L_1 and L_2 with associated bijection τ and scaling h . Then the Dirichlet forms satisfy $D(Q_2) = UD(Q_1)$ and there exists some constant $\beta > 0$, such that*

$$Q_1(f, g) = \beta Q_2(Uf, Ug),$$

for each $f, g \in D(Q_1)$. Furthermore, the equalities

$$\begin{aligned} m_1(\tau(w)) &= \beta h(w)^2 m_2(w) \\ b_1(\tau(x), \tau(y)) &= \beta h(x)h(y)b_2(x, y) \\ c_1(\tau(z)) &= \beta h(z)\mathcal{L}_2 h(z), \end{aligned}$$

hold for all $w, x, y, z \in X_2$. Here \mathcal{L}_2 is the formal operator associated with (b_2, c_2) .

Proof. By Corollary 3.5 there exists some $\beta > 0$ such that the map U satisfies

$$UU^* = \frac{1}{\beta} \text{Id}_2 \text{ and } U^*U = \frac{1}{\beta} \text{Id}_1,$$

where Id_i are the identity mappings on $\ell^2(X_i, m_i)$, $i = 1, 2$. Furthermore, for this β the proof of Lemma 3.4 shows $m_1 \circ \tau = \beta h^2 m_2$.

We compute for $f \in D(Q_1)$

$$\begin{aligned} Q_1(f, f) &= \lim_{t \rightarrow 0} \frac{1}{t} \langle f - e^{-tL_1} f, f \rangle \\ &= \beta \lim_{t \rightarrow 0} \frac{1}{t} \langle U^*Uf - e^{-tL_1} U^*Uf, f \rangle \\ &= \beta \lim_{t \rightarrow 0} \frac{1}{t} \langle Uf - e^{-tL_2} Uf, Uf \rangle \\ &= \beta Q_2(Uf, Uf). \end{aligned}$$

This shows $Uf \in D(Q_2)$ and, therefore, $UD(Q_1) \subseteq D(Q_2)$. A similar computation with $U^{-1} = \beta U^*$ yields $UD(Q_1) = D(Q_2)$. The formula $Q_1(f, g) = \beta Q_2(Uf, Ug)$ follows by polarization. For the statement on the graph structures we note that a short computation using Green's formula shows

$$Q_i(1_x, 1_y) = \mathcal{L}_i 1_x(y) = \begin{cases} -b_i(x, y), & \text{if } x \neq y, \\ \sum_z b_i(x, z) + c_i(x), & \text{if } x = y, \end{cases}$$

for $i = 1, 2$. Thus, for $x, y \in X_2$ we may compute

$$\begin{aligned} b_1(\tau(x), \tau(y)) &= -Q_1(1_{\tau(x)}, 1_{\tau(y)}) \\ &= -\beta Q_2(U1_{\tau(x)}, U1_{\tau(y)}) \\ &= -\beta h(x)h(y)Q_2(1_x, 1_y) \\ &= \beta h(x)h(y)b_2(x, y). \end{aligned}$$

Using this equality, we obtain for $z \in X_2$

$$\begin{aligned} c_1(\tau(z)) &= Q_1(1_{\tau(z)}) - \sum_{w \in X_1} b_1(\tau(z), w) \\ &= \beta \left(h(z)^2 Q_2(1_z) - \sum_{v \in X_2} h(z)h(v)b_2(z, v) \right) \\ &= \beta h(z) \left(\sum_{v \in X_2} b(z, v)h(z) + c_2(z)h(z) - \sum_{v \in X_2} h(v)b_2(z, v) \right) \\ &= \beta h(z)\mathcal{L}_2 h(z). \end{aligned}$$

This finishes the proof. \square

A relevant consequence of the previous theorem is the following.

Corollary 3.7. *For $i = 1, 2$, let Q_i be a Dirichlet form associated to the graph (b_i, c_i) over (X_i, m_i) . Let $U: \ell^2(X_1, m_1) \rightarrow \ell^2(X_2, m_2)$ be an order isomorphism intertwining L_1 and L_2 with associated bijection τ and scaling h . Then*

$$\text{Deg}_1(\tau(x)) = \text{Deg}_2(x),$$

for all $x \in X_2$.

Proof. Using the previous theorem we may compute

$$\begin{aligned} \text{Deg}_1(\tau(x)) &= \frac{1}{m_1(\tau(x))} \left(\sum_y b_1(\tau(x), y) + c_1(\tau(x)) \right) \\ &= \frac{1}{m_1(\tau(x))} Q_1(1_{\tau(x)}, 1_{\tau(x)}) \\ &= \frac{\beta h(x)^2}{m_1(\tau(x))} Q_2(1_x, 1_x) \\ &= \frac{1}{m_2(x)} Q_2(1_x, 1_x) \\ &= \text{Deg}_2(x). \end{aligned} \quad \square$$

The statement of the previous theorem can be substantially strengthened if more information on the measures m_1, m_2 and / or the function h is available. This is the content of later sections. Here, we derive next a special feature of h . We need one more definition.

Definition 3.8 ((Super)Harmonic functions). Let (b, c) be a graph over X with associated formal Laplacian \mathcal{L} . A function $f: X \rightarrow \mathbb{R}$ is called *superharmonic* if $f \in \mathcal{F}$ and $\mathcal{L}f \geq 0$. It is called *harmonic* if $f \in \mathcal{F}$ and $\mathcal{L}f = 0$.

Remark. Note that a function f is harmonic if and only if $\tilde{L}f = 0$ for any operator \tilde{L} of the form $\tilde{L} = \frac{1}{m}\mathcal{L}$ with $m: X \rightarrow (0, \infty)$.

Corollary 3.9. *For $i = 1, 2$, let Q_i be a Dirichlet form associated to the graph (b_i, c_i) over (X_i, m_i) . Let $U: \ell^2(X_1, m_1) \rightarrow \ell^2(X_2, m_2)$ be an order isomorphism intertwining L_1 and L_2 . Then the scaling h associated to U is superharmonic. Furthermore, h is harmonic if and only if $c_1 \equiv 0$.*

Proof. As h is strictly positive, this is an immediate consequence of the formula

$$c_1(\tau(x)) = \beta h(x) \mathcal{L}_2 h(x). \quad \square$$

Remark. In this section, we have assumed connectedness of the graphs in some places. If this condition is dropped the statements still remain valid on connected components of the graph. Thus, all our (subsequent) results will hold true on connected components. In this sense, connectedness can always be replaced with the statement being true on connected components.

4. A GENERALIZED GROUND STATE TRANSFORM

In this section we prove a converse of Theorem 3.6 for regular forms, which will show that our result is optimal if we do not impose further restrictions on the underlying graph structures. For this purpose we introduce the notion of generalized ground state transforms with respect to a superharmonic function.

Consider a graph (b, c) over (X, m) with associated regular Dirichlet form $Q^{(D)}$. Furthermore, suppose h is a strictly positive superharmonic function, $\tau : X \rightarrow X$ is a bijection and $\beta > 0$. We define the graph (b_h, c_h) over (X, m_h) by setting

$$\begin{aligned} m_h(\tau(w)) &= \beta h(w)^2 m(w), \\ b_h(\tau(x), \tau(y)) &= \beta h(x)h(y)b(x, y), \\ c_h(\tau(z)) &= \beta h(z)\mathcal{L}h(z), \end{aligned}$$

hold for all $w, x, y, z \in X$.

Definition 4.1. The regular Dirichlet form $Q_h = Q_{h, \tau, \beta}$ associated with (b_h, c_h) over (X, m_h) is called *generalized ground state transform* of $Q^{(D)}$ with respect to triplet (h, τ, β) . Its associated operator is denoted by L_h .

Remark. If h is harmonic, $\tau \equiv \text{id}$ and $\beta \equiv 1$ the above construction is known as ground state transform of $Q^{(D)}$, see e.g. [16, 23] for more discussions on the topic.

Lemma 4.2. Let (b, c) be a graph over (X, m) and let $Q_{h, \tau, \beta}$ be a generalized ground state transform of the corresponding regular Dirichlet form $Q^{(D)}$. Then, the order isomorphism $U : \ell^2(X, m_h) \rightarrow \ell^2(X, m)$, $f \mapsto h \cdot (f \circ \tau)$ satisfies $UD(Q_h) = D(Q^{(D)})$ and

$$Q_h(f, g) = \beta Q^{(D)}(Uf, Ug),$$

for each $f, g \in D(Q_h)$.

Proof. For $f \in \ell^2(X, m_h)$ the definition of m_h yields

$$\beta \|Uf\|_m^2 = \|f\|_{m_h}^2.$$

For $x \in X$ we can compute

$$\begin{aligned} Q_h(1_x, 1_x) &= \sum_{y \in X} b_h(x, y) + c_h(x) \\ &= \beta \sum_{y \in X} h(\tau^{-1}(x))h(y)b(\tau^{-1}(x), y) + \beta h(\tau^{-1}(x))\mathcal{L}h(\tau^{-1}(x)) \\ &= \beta h(\tau^{-1}(x))^2 \sum_{y \in X} b(\tau^{-1}(x), y) + \beta h(\tau^{-1}(x))^2 c(\tau^{-1}(x)) \\ &= \beta Q^{(D)}(U1_x, U1_x). \end{aligned}$$

For two distinct $x, y \in X$, we obtain

$$\begin{aligned} \beta Q(U1_x, U1_y) &= -\beta h(\tau^{-1}(x))h(\tau^{-1}(y))b(\tau^{-1}(x), \tau^{-1}(y)) \\ &= -b_h(x, y) = Q_h(1_x, 1_y). \end{aligned}$$

Now the bilinearity of $Q^{(D)}$ and Q_h as well as the denseness of $C_c(X)$ in the form domains show the statement. \square

Theorem 4.3. *Let (b, c) be a graph over (X, m) . Let $Q^{(D)}$ be the corresponding regular Dirichlet form with associated operator $L^{(D)}$ and let $Q_{h,\tau,\beta}$ be a generalized ground state transform of $Q^{(D)}$. Then, the order isomorphism $U : \ell^2(X, m_h) \rightarrow \ell^2(X, m)$, $f \mapsto h \cdot (f \circ \tau)$ intertwines L_h and $L^{(D)}$.*

Proof. From the definitions it is obvious that $UU^* = \frac{1}{\beta}\text{Id}$. Using the previous lemma we obtain for $f \in D(L_h)$ and $g \in D(Q^{(D)})$

$$\begin{aligned} \langle UL_h f, g \rangle &= \langle L_h f, U^* g \rangle \\ &= Q_h(f, U^* g) \\ &= \beta Q(Uf, UU^* g) \\ &= Q(Uf, g). \end{aligned}$$

This shows $Uf \in D(L^{(D)})$ and $L^{(D)}Uf = UL_h f$. A similar argument with U^{-1} yields the inclusion $D(L^{(D)}) \subseteq D(L_h)$. \square

Remark. The previous theorem shows that the graph structure of a regular Dirichlet form is determined by its diffusion up to a generalized ground state transform. In particular, if the graph (b, c) admits a strictly positive non-constant superharmonic function diffusion does not uniquely determine the graph.

5. THE MAP τ AS AN ISOMETRIC GRAPH ISOMORPHISM

In this section we use the material presented so far to obtain some results on how much diffusion determines the graph in the general case. More specifically, the results below will show that graphs with Markovian semigroups which are equal up to order isomorphism are isometric both with respect to combinatorial graph metric d and to the intrinsic metric ϱ defined via the generalized degree.

Theorem 5.1 (τ as isometry w.r.t. d). *For $i = 1, 2$, let Q_i be a Dirichlet form associated to the graph (b_i, c_i) over (X_i, m_i) . Let $U : \ell^2(X_1, m_1) \rightarrow \ell^2(X_2, m_2)$ be an order isomorphism intertwining L_1 and L_2 with associated bijection τ . Then, $b_1(\tau(y), \tau(y')) > 0$ if and only if $b_2(y, y') > 0$ holds. In particular, (b_1, c_1) is connected if and only if (b_2, c_2) is connected and τ is an isometry with respect to the combinatorial graph metric in this case.*

Proof. By Theorem 3.6 we have

$$b_1(\tau(y), \tau(z)) = \beta h(y)h(z)b_2(y, z)$$

for all y, z in the same connected component of X_2 . Now our first claim follows because h, β are strictly positive.

Moreover, the equation shows that (y_0, \dots, y_n) is a path in (X_2, b_2, c_2) if and only if $(\tau(y_0), \dots, \tau(y_n))$ is a path in (X_1, b_1, c_1) which implies that (X_1, b_1, c_1) is connected if and only if (X_2, b_2, c_2) is connected as well as $d_1(\tau(y), \tau(z)) = d_2(y, z)$ for all $y, z \in X_2$. \square

The above theorem shows that graphs with semigroups that are equivalent via an intertwining order isomorphism have the same combinatorial structure.

We can also obtain the following variant of Theorem 5.1.

Theorem 5.2 (τ as isometry w.r.t. ϱ). *For $i = 1, 2$, let Q_i be a Dirichlet form associated to the connected graph (b_i, c_i) over (X_i, m_i) . Furthermore, let $U: \ell^2(X_1, m_1) \rightarrow \ell^2(X_2, m_2)$ be an order isomorphism intertwining L_1 and L_2 with associated bijection τ . Then τ is isometric with respect to the pseudo metrics ϱ_1, ϱ_2 .*

Proof. This follows immediately from Corollary 3.7 and Theorem 3.6 (compare proof of Theorem 5.1 as well) since ϱ_1 and ϱ_2 only depend on the combinatorial structure (paths in the graph) and the vertex degrees. \square

6. RECURRENT GRAPHS

In this section we will study a class of graphs with the property that every positive (super)harmonic function must be constant. This will allow us to give a generalization of Theorem 5.1. The results of this section are essentially all known. For the convenience of the reader we include some proofs.

Definition 6.1 (Recurrence and transience). A Markovian semigroup $(e^{-tL})_{t \geq 0}$ on $\ell^2(X, m)$ is called recurrent if

$$G(x, y) := \int_0^\infty e^{-tL} 1_x(y) dt = \infty$$

for all $x, y \in X$. It is called transient if $G(x, y) < \infty$ for all $x, y \in X$. A graph over (X, m) is called recurrent/transient if the semigroup $(e^{-tL^{(D)}})_{t \geq 0}$ of the associated Dirichlet Laplacian $L^{(D)}$ is recurrent/transient.

For a connected graph, the semigroup $(e^{-tL^{(D)}})_{t \geq 0}$ is irreducible (i.e., the semigroup maps non-negative, non-zero functions to strictly positive ones) by [24] and, hence, the graph is either recurrent or transient [17] (see [49], Proposition 3.2) as well.

The following lemma is rather easy to prove and well-known in an even more general context [17]. Thus, we refrain from giving a proof.

Lemma 6.2. *A Markovian semigroup $(e^{-tL})_{t \geq 0}$ on $\ell^2(X, m)$ is recurrent if and only if*

$$(Gf)(y) := \int_0^\infty e^{-tL} f(y) dt = \infty$$

for all $f \in \ell^2(X, m)$ with $f \geq 0$ and $f \neq 0$ and all $y \in X$.

As a corollary we can easily obtain that recurrence is stable under order isomorphisms. Again, we omit the simple proof.

Corollary 6.3. *For $i = 1, 2$, let Q_i be a Dirichlet form associated to the graph (b_i, c_i) over (X_i, m_i) and let L_i be the corresponding Laplacian. Let $U: \ell^2(X_1, m_1) \rightarrow \ell^2(X_2, m_2)$ be an order isomorphism intertwining the*

Laplacians. Then, $(e^{-tL_1})_{t \geq 0}$ is recurrent if and only if $(e^{-tL_2})_{t \geq 0}$ is recurrent.

For later use we recall the following criterion for recurrence (see e.g. [17, 50]).

Theorem 6.4. *Let (b, c) be a connected graph over (X, m) . Choose $o \in X$ arbitrary and define the norm $\|\cdot\|_o$ on \mathcal{D} by*

$$\|f\|_o = \sqrt{\mathcal{Q}(f, f) + |f(o)|^2}.$$

Then, the following assertions are equivalent:

- (i) *The graph is recurrent.*
- (ii) *The closure of $C_c(X)$ with respect to $\|\cdot\|_o$ contains the constant function 1 and $\mathcal{Q}(1, 1) = 0$.*

Remark. Note that the theorem gives, in particular, that recurrence does not depend on the choice of the measure m . Hence, from now on, we will call a graph recurrent/transient without referring to the measure m .

The previous theorem yields the following lemma (cf. [49], Proposition 3.12) which justifies that we will assume $c = 0$ in the following.

Lemma 6.5. *If (b, c) is a connected graph over (X, m) with $c \neq 0$ then it is transient.*

From Theorem 6.4, we obtain the following characterization of recurrence (see [54], Theorem 6.21 as well).

Proposition 6.6. *Let $(b, 0)$ be a connected graph over (X, m) . Then the following assertions are equivalent:*

- (i) *The graph is recurrent.*
- (ii) *All positive superharmonic functions are constant.*

Proof. (i) \implies (ii): Let \mathcal{Q} be the generalized form associated to the graph $(b, 0)$ and let h be a positive superharmonic function. As $(b, 0)$ is connected, h is strictly positive (see e.g. Proposition 3.4 in [23]). We denote by \mathcal{Q}_h the generalized form corresponding to the graph of the ground state transform (b_h, c_h) which was introduced in Section 4. Then, Lemma 4.2 yields

$$\mathcal{Q}_h\left(\frac{\varphi}{h}, \frac{\varphi}{h}\right) = \mathcal{Q}(\varphi, \varphi)$$

for all $\varphi \in C_c(X)$. By Theorem 6.4, we can now find a sequence (φ_n) in $C_c(X)$ converging to 1 with respect to $\|\cdot\|_o$. As discussed in [50] (see e.g. [19] as well) convergence with respect to $\|\cdot\|_o$ implies pointwise convergence. Thus, we can apply the Fatou lemma to the above equality between \mathcal{Q}_u and \mathcal{Q} to obtain that

$$0 \leq \mathcal{Q}_h\left(\frac{1}{h}, \frac{1}{h}\right) = \mathcal{Q}(1, 1) = 0.$$

By connectedness of the graph, this gives that $\frac{1}{h}$ is constant. Hence, h is constant.

(ii) \implies (i): Assume $(b, 0)$ is not recurrent. We let $\mathcal{D}_o = \overline{C_c(X)}^{\|\cdot\|_o}$. By standard Hilbert space arguments there exists a unique minimizer h of the quadratic functional $\|\cdot\|_o^2$ on the closed convex set $\{u \in \mathcal{D}_o : u(o) \geq 1\}$. As

$\|(0 \vee h) \wedge 1\|_o \leq \|h\|_o$, the function h is positive and satisfies $h(o) = 1$. We now show that h is superharmonic. For $x \in X$ and $\varepsilon > 0$, we obtain

$$\begin{aligned} \mathcal{Q}(h, h) &= \|h\|_o - 1 \\ &\leq \|(h + \varepsilon 1_x) \wedge 1\|_o - 1 \\ &= \mathcal{Q}((h + \varepsilon 1_x) \wedge 1, (h + \varepsilon 1_x) \wedge 1) \\ &\leq \mathcal{Q}(h + \varepsilon 1_x, h + \varepsilon 1_x) \\ &= \mathcal{Q}(h, h) + 2\varepsilon \mathcal{L}h(x) + \varepsilon^2 \mathcal{Q}(1_x, 1_x). \end{aligned}$$

As this holds for all $\varepsilon > 0$, we infer $\mathcal{L}h(x) \geq 0$ for all $x \in X$. As the graph is not recurrent the constant functions do not belong to \mathcal{D}_o , hence, h is not constant. \square

Finally, we note the following (see [42], Theorem 6.2 as well).

Proposition 6.7. *If the graph $(b, 0)$ over (X, m) is recurrent, then the associated Dirichlet forms $Q^{(D)}$ and $Q^{(N)}$ agree. In particular, there is then only one Dirichlet form associated to the graph.*

Proof. Assume $(b, 0)$ is recurrent. Let (φ_n) be a sequence in $C_c(X)$ converging to 1 with respect to $\|\cdot\|_o$. By the cut-off properties of \mathcal{Q} this sequence can be chosen to satisfy $0 \leq \varphi_n \leq 1$. Let $f \in D(Q^{(N)})$ be bounded. Then $\psi_n := f \cdot \varphi_n \in D(Q^{(D)})$ for each n . Furthermore, as $Q^{(N)}$ is a Dirichlet form, we obtain from Theorem 1.4.2. of [17]

$$Q^{(D)}(\psi_n, \psi_n) = Q^{(N)}(\psi_n, \psi_n) \leq \|f\|_\infty \sqrt{Q^{(N)}(\varphi_n)} + \|\varphi_n\|_\infty \sqrt{Q^{(N)}(f)}.$$

This shows that (ψ_n) is bounded with respect to the form norm $\|\cdot\|_Q$. Thus, (ψ_n) has a weakly convergent subsequence with limit $\psi \in D(Q^{(D)})$. As $\psi_n \rightarrow f$ pointwise, we infer $f = \psi$. Every function in $D(Q^{(N)})$ can be approximated by bounded functions, hence, the claim follows. \square

The previous proposition allows us to speak about *the* Dirichlet form associated to $(b, 0)$ over (X, m) whenever the graph $(b, 0)$ is recurrent.

7. DIFFUSION DETERMINES THE RECURRENT GRAPH

In this section we show that a recurrent graph is completely determined by the diffusion associated to it (up to an overall constant). As we will see the condition of recurrence is necessary and the appearance of the overall constant can not be avoided.

Recall that recurrence implies the uniqueness of the associated Dirichlet form.

Theorem 7.1. *Let $(b_1, 0)$ be a connected recurrent graph over (X_1, m_1) and Q_1 the associated Dirichlet form and L_1 its generator. Let Q_2 be a Dirichlet form associated to a graph $(b_2, 0)$ over (X_2, m_2) and L_2 its generator. Let $U: \ell^2(X_1, m_1) \rightarrow \ell^2(X_2, m_2)$ be an order isomorphism intertwining L_1 and L_2 and τ the associated bijection and h the associated scaling. Then there*

is a constant $\beta > 0$ such that

$$\begin{aligned} b_1(\tau(y), \tau(z)) &= \beta b_2(y, z), \\ m_1(\tau(y)) &= \beta m_2(y) \end{aligned}$$

hold for all $y, z \in X_2$.

Proof. By Lemma 3.9, the function h is harmonic. As $(b_2, 0)$ is also recurrent by Corollary 6.3, the function h is constant by Proposition 6.6 and the statement follows from Theorem 3.6. \square

Remark. The theorem is optimal in the following sense: The condition of recurrence (and hence of vanishing c) is necessary. If a graph is not recurrent then, as seen in the previous section, it admits a positive non-constant superharmonic function. As shown by the discussion in Section 4 such superharmonic functions lead to intertwined operators on different graph structures. Similarly, the appearance of the constant β can not be avoided in the general case as one could do a generalized ground state transform with the constant function 1 and some normalizing constant $\beta > 0$.

From the theorem we also obtain the following characterization.

Corollary 7.2. *For $i = 1, 2$, let Q_i be a Dirichlet form associated to the graph $(b_i, 0)$ over (X_i, m_i) and let L_i be the corresponding Laplacian. Assume that both graphs are recurrent and connected. Then, the following assertions are equivalent:*

- (i) *There is an order isomorphism $U: \ell^2(X_1, m_1) \rightarrow \ell^2(X_2, m_2)$ intertwining L_1 and L_2 .*
- (ii) *There is a bijection $\tau: X_2 \rightarrow X_1$ and a constant $\beta > 0$ such that*

$$\begin{aligned} b_1(\tau(y), \tau(z)) &= \beta b_2(y, z) \\ m_1(\tau(y)) &= \beta m_2(y) \end{aligned}$$

for all $y, z \in X_2$ and L_2 is the Dirichlet (resp. Neumann) Laplacian on the graph $(b_2, 0)$ over (X_2, m_2) .

Proof. (i) \Rightarrow (ii) is a consequence of the previous theorem. (ii) \Rightarrow (i) follows from the discussion about the generalized ground state transform, Theorem 4.3 and the fact that recurrence implies the regularity of Q_i , $i = 1, 2$, Proposition 6.7. \square

Remark. Of course, if X is finite and c vanishes the recurrence assumption is automatically satisfied. This can be substantially strengthened as discussed in the next section.

8. APPLICATION TO GRAPHS WITH FINITE TOTAL EDGE WEIGHT

As noted in the previous section graphs over finite sets are determined by their diffusion. Of course, any measure on a finite set is finite and the Laplacian associated to a graph over a finite set is bounded. It turns out that these two requirements alone already imply that the diffusion determines the graph. In fact, these two requirements imply finiteness of the total edge weight, which in turn implies recurrence. As a consequence we obtain that the diffusion determines the graph whenever the total edge weight is finite,

Corollary 8.5. These results can be considered as rather complete analogues of [2, 5] in the sense that the (relative) compactness assumption there is replaced by finiteness of the measure and boundedness of the Laplacian.

We start with a characterization of boundedness of the Laplacian taken from [24].

Recall that the *generalized degree* $\text{Deg} : X \rightarrow [0, \infty)$ of a graph (b, c) over (X, m) is defined by

$$\text{Deg}(x) := \frac{1}{m(x)} \left(\sum_{y \in X} b(x, y) + c(x) \right), \quad x \in X.$$

Proposition 8.1 (Characterization of boundedness [24]). *Let (b, c) be a graph over (X, m) . Then, the following assertions are equivalent:*

- (i) *There exists a $C > 0$ such that $\text{Deg}(x) \leq C$ holds for all $x \in X$.*
- (ii) *There exists a $c > 0$ such that $\mathcal{Q}(f, f) \leq c \|f\|^2$ for all $f \in \ell^2(X, m)$.*
- (iii) *The operator \tilde{L} restricts to a bounded operator on $\ell^2(X, m)$.*
- (iv) *The operator \tilde{L} restricts to a bounded operator on $\ell^p(X, m)$ for all $p \in [1, \infty]$.*

Definition 8.2. In the situation of the proposition we say that the graph (b, c) over (X, m) has a *bounded Laplacian*.

Furthermore, for a graph (b, c) over X recall the *normalizing measure* $n : X \rightarrow (0, \infty)$ given by

$$n(x) = \sum_{y \in X} b(x, y) + c(x), \quad x \in X.$$

Example. (Normalized Laplacian and normalized form) A special instance of a bounded Laplacian is the *normalized Laplacian* associated to the graph $(b, 0)$ over X . This Laplacian arises by the choice of the measure $m = n$ such that the generalized degree defined above equals one. This normalized Laplacian is heavily studied (see discussion in the introduction). We will refer to the induced Markovian semigroup as normalized diffusion on the graph $(b, 0)$.

In the case of finite X , the associated Laplacian is bounded and the total mass is finite. It is possible to characterize the occurrence of these two features for general countable X via the *total edge weight*

$$n(X) = \sum_{x, y \in X} b(x, y) + \sum_{x \in X} c(x).$$

This is the content of the next proposition.

Proposition 8.3. *Let (b, c) be a graph over X . Then, the following two assertions are equivalent:*

- (i) *There exists a measure m on X with finite total mass such that the graph (b, c) over (X, m) has a bounded Laplacian.*
- (ii) *The total edge weight*

$$n(X) = \sum_{x, y \in X} b(x, y) + \sum_{x \in X} c(x)$$

is finite.

Proof. (i) \implies (ii): Let m be such a measure. By the first proposition of this section, there exists then a $C \geq 0$ with $\text{Deg} \leq C$. A short computation now gives

$$\sum_{x,y \in X} b(x,y) + \sum_{x \in X} c(x) = \sum_{x \in X} \text{Deg}(x)m(x) \leq Cm(X) < \infty.$$

(ii) \implies (i): Consider the measure $m = n$. Then, m has finite total mass by (ii) and the associated Laplacian is bounded. \square

As is certainly well-known any graph with finite total edge weight is recurrent. We include a proof for completeness reasons.

Proposition 8.4. *Let $(b, 0)$ be a connected graph over X with finite total edge weight. Then, the graph is recurrent.*

Proof. This follows easily from Theorem 6.4. In fact, any sequence (φ_n) in $C_c(X)$ with $0 \leq \varphi_n \leq 2$ and $\varphi_n(x) \rightarrow 1$, $n \rightarrow \infty$, for all $x \in X$ will converge to 1 with respect to $\|\cdot\|_o$ due to the finiteness assumption on the total edge weight. \square

From the previous proposition and Theorem 7.1 we immediately obtain the following corollary.

Corollary 8.5 (Diffusion determines the graph of total finite edge weight). *Let $(b_1, 0)$ be a connected graph over (X_1, m_1) with*

$$\sum_{x,y} b_1(x,y) < \infty.$$

Let Q_1 be the associated Dirichlet form with associated Laplacian L_1 . Let Q_2 be a Dirichlet form associated to the graph $(b_2, 0)$ over (X_2, m_2) and let L_2 be its generator. Let $U : \ell^2(X_1, m_1) \rightarrow \ell^2(X_2, m_2)$ be an order isomorphism intertwining L_1 and L_2 with associated bijection τ and associated scaling h . Then there is a constant $\beta > 0$ such that

$$\begin{aligned} b_1(\tau(y), \tau(z)) &= \beta b_2(y, z), \\ m_1(\tau(y)) &= \beta m_2(y) \end{aligned}$$

hold for all $y, z \in X_2$.

Remark. As discussed in the introduction to this subsection, the previous theorem generalizes the case of finite X and can be seen to provide an analogue to the results of [2, 5].

9. TWO SPECIAL SITUATIONS

In this section, we have a look at two special situations, where we can say more even without a recurrence condition. These two situations are the case of constant measure and the case of a graph with standard weights and normalized Laplacian (see discussion in the introduction).

The case $m = 1$. If the measure is normalized to be equal to one in each point, we obtain the following theorem.

Theorem 9.1. *For $i = 1, 2$, let Q_i be a Dirichlet form associated to the graph (b_i, c_i) over (X_i, m_i) . Assume that (b_1, c_1) is connected and $m_1 \equiv 1$ and $m_2 \equiv 1$. Let $U : \ell^2(X_1, m_1) \rightarrow \ell^2(X_2, m_2)$ be an order isomorphism intertwining L_1 and L_2 with associated bijection τ . Then the equalities $b_1(\tau(y), \tau(z)) = b_2(y, z)$ and $c_1(\tau(y)) = c_2(y)$ hold for all $y, z \in X_2$.*

Proof. By Theorem 3.6 there exists a constant $\beta > 0$ such that

$$m_1(\tau(x)) = \beta h(x)^2 m_2(x).$$

Using $m_1 \equiv 1$ and $m_2 \equiv 1$ we obtain $h(x) = \frac{1}{\sqrt{\beta}}$ for all $x \in X_2$. Now the statement follows from the other formulas of Theorem 3.6. \square

Remark. As the proof shows, it suffices to assume that τ is measure preserving, that is, $m_2(y) = m_1(\tau(y))$ for all $y \in X_2$.

The normalized Laplacian for graphs with standard weights.

Here, we consider the situation that $(b, 0)$ is a graph over X with b taking values in $\{0, 1\}$ and the measure n given by $n(x) = \sum_y b(x, y)$. Thus, we consider the normalized Laplacian on a graph with standard weights. In this situation, the associated Dirichlet form is bounded, Proposition 8.1. In particular, it is the unique form associated to the graph $(b, 0)$ over (X, n) . From Theorem 5.1 we immediately infer the following result.

Theorem 9.2. *For $i = 1, 2$, let $(b_i, 0)$ be a graph over X_i with b_i taking values in $\{0, 1\}$ and define the measure n_i on X_i by $n_i(x) = \sum_{y \in X_i} b_i(x, y)$ and let Q_i be the associated Dirichlet form. Suppose $U : \ell^2(X_1, n_1) \rightarrow \ell^2(X_2, n_2)$ is an order isomorphism intertwining L_1 and L_2 with associated bijection τ . Then, the equality $b_1(\tau(y), \tau(z)) = b_2(y, z)$ holds for all $y, z \in X_2$.*

Proof. As both b_1 and b_2 take values only in $\{0, 1\}$ the equality

$$b_1(\tau(y), \tau(z)) = b_2(y, z),$$

for $y, z \in X_2$ follows directly from Theorem 5.1. \square

Remark. In the situation of the previous Theorem it is even possible to include non-vanishing c 's in the graphs. One just has to convince oneself that one still has unique forms associated to the arising graphs (b_i, c_i) due to the boundedness of the forms induced by $(b_i, 0)$, $i = 1, 2$.

10. REMARK ON THE CASE OF DISCRETE TIME

Instead of studying intertwining of continuous time semigroups which describe the time evolution of a continuous time Markov chain we can also consider their discrete time counterpart. The raised question then becomes: 'Do discrete time Markov chains determine the graph?' This is briefly discussed in this section.

For a graph $(b, 0)$ over the vertex set X , we consider the normalizing measure $n : X \rightarrow (0, \infty)$ given by

$$n(x) = \sum_{y \in X} b(x, y)$$

and the Markov operator $P : \mathcal{F} \rightarrow C(X)$ by setting

$$Pf(x) = \frac{1}{n(x)} \sum_{y \in V} b(x, y) f(y).$$

The choice of the measure n ensures that P is a bounded self-adjoint operator when restricted to $\ell^2(X, n)$. The powers of the Markov operator form a discrete time semigroup which describes the time evolution of the Markov chain $(X_n)_{n \geq 1}$ with transition probabilities

$$\mathbb{P}(X_n = x \mid X_{n-1} = y) = \frac{b(x, y)}{n(x)}.$$

Note that all powers of two Markov operators are intertwined by an order isomorphism if and only if the Markov operators itself are intertwined. Thus, the analogue of Theorem 3.6 in discrete time reads as follows.

Theorem 10.1. *For $i = 1, 2$, let $(b_i, 0)$ be a graph over X_i with corresponding Markov operator P_i and let $(b_1, 0)$ be connected. Furthermore, let $U : \ell^2(X_1, n_1) \rightarrow \ell^2(X_2, n_2)$ be an order isomorphism with associated bijection τ and scaling h , such that $UP_1 = P_2U$. Then there exists a constant $\beta > 0$, such that the equalities*

$$\begin{aligned} n_1(\tau(w)) &= \beta h(w)^2 n_2(w), \\ b_1(\tau(x), \tau(y)) &= \beta h(x) h(y) b_2(x, y), \\ P_2 h(z) &= h(z), \end{aligned}$$

hold for all $w, x, y, z \in X_2$.

Proof. Let $Q_i^{(D)}$ be the regular Dirichlet form associated with the graph $(b_i, 0)$ over (X_i, n_i) . Then, the associated operator L_i is given by

$$L_i f(x) = \frac{1}{n_i(x)} \sum_{y \in X_i} b_i(x, y) (f(x) - f(y)) = f(x) - P_i f(x).$$

This shows that the L_i are bounded and satisfy $UL_1 = L_2U$. Thus, the statement follows from Theorem 3.6 and the observation that $P_2 h = h$ is equivalent to $\mathcal{L}_2 h = 0$. \square

As in the continuous time setting we can strengthen this theorem for recurrent graphs.

Theorem 10.2. *For $i = 1, 2$, let $(b_i, 0)$ be a graph over X_i with corresponding Markov operator P_i and let $(b_1, 0)$ be recurrent and connected. Furthermore, let $U : \ell^2(X_1, n_1) \rightarrow \ell^2(X_2, n_2)$ be an order isomorphism with associated bijection τ and scaling h , such that $UP_1 = P_2U$. Then there exists a constant $\beta > 0$, such that the equalities*

$$\begin{aligned} n_1(\tau(w)) &= \beta n_2(w), \\ b_1(\tau(x), \tau(y)) &= \beta b_2(x, y), \end{aligned}$$

hold for all $w, x, y \in X_2$.

Proof. As $P_2 h = h$ if and only if $\mathcal{L}_2 h = 0$, the function h is harmonic by Theorem 10.1. The proof of Theorem 10.1 shows that $UP_1 = P_2U$ implies intertwining of the continuous time semigroups associated with $(b_i, 0)$ over

$(X_i, n_i), i = 1, 2$. Therefore, Corollary 6.3 shows that $(b_2, 0)$ is also recurrent. Thus, the function h is constant by Proposition 6.6 and the statement follows from Theorem 10.1. \square

Remark. In Section 6 we defined recurrence via properties of continuous time Markovian semigroups and then observed that this definition is independent of the underlying measure m . However, this may seem artificial in the discrete time setting where one usually uses a different definition for recurrence. Namely, one requires the sum

$$\sum_{n \geq 1} P^n 1_x(y)$$

to diverge for all $x, y \in X$. It is well known that the divergence of this series can be characterized by positive superharmonic functions being constant (see e.g. [54]). Thus, Proposition 6.6 shows that these two notions of recurrence agree.

APPENDIX A. A PROPOSITION ON INTERTWINING

In this section we discuss a characterization on intertwining. Such a characterization is well-known in many contexts. It is certainly also known in our context. As we could not find it in the literature, we include a discussion.

Proposition A.1. *Let B_1, B_2 be Banach spaces and let $S_1(t), t \geq 0$, and $S_2(t), t \geq 0$, strongly continuous semigroups with generators $-L_1$ and $-L_2$ respectively. Let $U \in \mathcal{L}(B_1, B_2)$ invertible. Then, the following assertions are equivalent:*

- (i) $US_1(t) = S_2(t)U$ for all $t \geq 0$.
- (ii) $UD(L_1) = D(L_2)$ and $UL_1f = L_2Uf$ for all $f \in D(L_1)$.

Proof. We write e^{-tL_i} for $S_i(t), i = 1, 2$.

(i) \implies (ii): The operator L_i is the generator of the semigroup $(e^{-tL_i})_{t \geq 0}$, $i \in \{1, 2\}$, hence

$$D(L_i) = \left\{ f \in B_i : \lim_{t \downarrow 0} \frac{1}{t} (f - e^{-tL_i} f) \text{ exists} \right\},$$

$$L_i f = \lim_{t \downarrow 0} \frac{1}{t} (f - e^{-tL_i} f).$$

Thus, we have (since U is a homeomorphism)

$$\begin{aligned} D(L_2) &= \left\{ f \in B_2 : \lim_{t \downarrow 0} \frac{1}{t} (f - Ue^{-tL_1}U^{-1}f) \text{ exists} \right\} \\ &= \left\{ f \in B_2 : \lim_{t \downarrow 0} U \frac{1}{t} (U^{-1}f - e^{-tL_1}U^{-1}f) \text{ exists} \right\} \\ &= \left\{ f \in B_2 : \lim_{t \downarrow 0} \frac{1}{t} (U^{-1}f - e^{-tL_1}U^{-1}f) \text{ exists} \right\} \\ &= UD(L_1) \end{aligned}$$

and for all $f \in B_1$

$$\begin{aligned} L_2 U f &= \lim_{t \downarrow 0} \frac{1}{t} (U f - e^{-t L_2} U f) \\ &= U \lim_{t \downarrow 0} \frac{1}{t} (f - e^{-t L_1} f) \\ &= U L_1 f. \end{aligned}$$

(ii) \implies (i): Let be $S_t = U e^{-t L_1} U^{-1}$. Then $(S_t)_{t \geq 0}$ is a strongly continuous semigroup on B_2

$$\begin{aligned} S_{t+s} &= U e^{-(t+s) L_1} U^{-1} = U e^{-t L_1} U^{-1} U e^{-s L_1} U^{-1} = S_t S_s \\ \lim_{t \rightarrow 0} U e^{-t L_1} U^{-1} f &= U \lim_{t \rightarrow 0} e^{-t L_1} U^{-1} f = U U^{-1} f = f \end{aligned}$$

for all $s, t \geq 0$, $f \in B_2$. The generator L of $(S_t)_{t \geq 0}$ is given by

$$\begin{aligned} D(L) &= \{f \in B_2 : \lim_{t \rightarrow 0} \frac{1}{t} (S_t f - f) \text{ exists}\} \\ &= \{f \in B_2 : \lim_{t \rightarrow 0} \frac{1}{t} (e^{-t L_1} U^{-1} f - U^{-1} f) \text{ exists}\} \\ &= \{f \in B_2 : U^{-1} f \in D(L_1)\} \\ &= U D(L_1) \\ &= D(L_2) \end{aligned}$$

and for all $f \in D(L) = D(L_2)$

$$\begin{aligned} L f &= \lim_{t \rightarrow 0} \frac{1}{t} (U e^{-t L_1} U^{-1} f - f) \\ &= U \lim_{t \rightarrow 0} \frac{1}{t} (e^{-t L_1} U^{-1} f - U^{-1} f) \\ &= U L_1 U^{-1} f \\ &= L_2 f. \end{aligned}$$

Since the generator determines the semigroup, we have $U e^{-t L_1} = e^{-t L_2} U$ for all $t \geq 0$. \square

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